

Inclusion theorems of double Deferred Cesàro means II

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Abstract

In 1932 R. P. Agnew present a definition for Deferred Cesàro mean. Using this definition R. P. Agnew present inclusion theorems for the deferred and none Deferred Cesàro means. This paper is part 2 of a series of papers that present extensions to the notion of double Deferred Cesàro means. Similar to part 1 this paper uses this definition and the notion of regularity for four dimensional matrices, to present extensions and variations of the inclusion theorems presented by R. P. Agnew in [2].

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1 Introduction

This paper is part 2 of a series of papers characterization the inclusion between Cesàro means and double Deferred Cesàro means. In part 1[11] we presented the notion of double Deferred Cesàro means which is a multi-dimensional analog and Agnew's Deferred Cesàro means in [2]. Using this notions and as series of basic results in [11], this paper present a series of inclusion theorems similar to the following: *The double Cesàro mean includes $D_{m-1, q_m, n-1, p_n}$ be a Deferred Cesàro mean with $q_m = m, p_n = n; m \neq \alpha_1, \alpha_2, \dots$ and $n \neq \beta_1, \beta_2, \dots$ with*

$$q_{\alpha_i} = \alpha_{i+1} - 1; i = 1, 2, 3, \dots, \alpha_m$$

and

$$p_{\beta_j} = \beta_{j+1} - 1; j = 1, 2, 3, \dots, \beta_n$$

where $\{q_{\alpha_i}\}$ and $\{p_{\beta_j}\}$ are increasing single dimensional sequences of integers such that $\alpha_m > m$ and $\beta_n > n$.

2 Definitions, notations and preliminary results

The definitions, notations, and preliminary results are similar to those in Part 1 [11] which are restated here for the purpose of completeness.

Definition 2.1 (Pringsheim, 1900). A double sequence $x = \{x_{k,l}\}$ has a **Pringsheim limit** L (denoted by $P\text{-lim } x = L$) provided that, given an $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. Such an $\{x\}$ is described more briefly as “ P -convergent”.

Definition 2.2 (Patterson, 2000). A double sequence $\{y\}$ is a **double subsequence** of $\{x\}$ provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that, if $\{x_j\} = \{x_{n_j, k_j}\}$, then $\{y\}$ is formed by

$$\begin{array}{cccc} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & - \end{array}$$

In [13] Robison presented the following notion of conservative four-dimensional matrix transformation and a Silverman-Toeplitz type characterization of such notion.

Definition 2.3. The four-dimensional matrix A is said to be **RH-regular** if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

The assumption of bounded was added because a double sequence which is P-convergent is not necessarily bounded. Along these same lines, Robison and Hamilton presented a Silverman-Toeplitz type multidimensional characterization of regularity in [3] and [13].

Theorem 2.4. (Hamilton [3], Robison [13]) The four-dimensional matrix A is RH-regular if and only if

$$\begin{aligned} RH_1: & \text{P-lim}_{m,n} a_{m,n,k,l} = 0 \text{ for each } k \text{ and } l; \\ RH_2: & \text{P-lim}_{m,n} \sum_{k,l=0,0}^{\infty, \infty} a_{m,n,k,l} = 1; \\ RH_3: & \text{P-lim}_{m,n} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0 \text{ for each } l; \\ RH_4: & \text{P-lim}_{m,n} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0 \text{ for each } k; \\ RH_5: & \sum_{k,l=0,0}^{\infty, \infty} |a_{m,n,k,l}| \text{ is P-convergent}; \\ RH_6: & \text{there exist finite positive integers } \Delta \text{ and } \Gamma \text{ such that} \\ & \sum_{k,l > \Gamma} |a_{m,n,k,l}| < \Delta. \end{aligned}$$

The main goals of this paper includes the comparison of double Cesàro mean transformation

$$(C, 1, 1)_{m,n,k,l} := \begin{cases} \frac{1}{mn}, & \text{if } k \leq m \text{ and } l \leq n \\ 0, & \text{if otherwise} \end{cases}$$

with the double Deferred Cesàro mean

$$D_{m,n,k,l} := \begin{cases} \frac{1}{(\alpha_m - \beta_m)(q_n - p_n)}, & \text{if } \beta_m < k \leq \alpha_m \text{ and } p_n < l \leq q_n, \\ 0, & \text{if otherwise} \end{cases}$$

where $[p_n]$ $[q_n]$ $[\alpha_m]$, and $[\beta_m]$ are sequences of nonnegative integers satisfying

$$\alpha_m < \beta_m, \text{ and } p_n < q_n \text{ for } m, n = 1, 2, \dots; \quad (1.1)$$

and

$$\lim_m \beta_m = +\infty, \text{ and } \lim_n q_n = +\infty. \quad (1.2)$$

Using these four dimensional transformations we shall present a catalog of inclusion theorems such as the following. *The four dimensional summability method M include $D_{p_n, \alpha_n, q_n, \beta_n}$ where p_n and q_n for almost all n is a give non-negative integer p if and only if α_n and β_n are almost all positive integers.*

3 Main results

Theorem 3.1. The Double Cesàro transformation includes every Double Deferred Cesàro mean of the form $D_{p_n, \alpha_n, q_n, \beta_n}$ for which α_n and β_n contains almost all positive integers.

Proof. Let $[x_{k,l}]$ be summable by $D_{p_n, \alpha_n, q_n, \beta_n}$ (say to L) such that $P\text{-}\lim_{m,n} D_{m,n} = L$ and choose two integers K and L large such that $[p_m]$ and $[q_n]$ contains all integers greater than K and L , respectively. Thus let $i_1 = i_2 = i_3 = \dots = i_K = 1$ and $j_1 = j_2 = j_3 = \dots = j_L = 1$ and determine for $m > K$ and $n > L$ index i_m and j_n is such that $p_{i_m} = m$ and $q_{j_n} = n$. Since $\lim_m i_m = +\infty$ and $\lim_n j_n = +\infty$, it follows

$$P\text{-}\lim_{m,n} D_{m,n} = L \text{ and } P\text{-}\lim_{m,n} D_{i_m, j_n} = L.$$

Therefore $[x]$ is summable by $D_{p_m, m, q_n, n}$ to L . The result follows from Lemma 3.3 of [11]. Q.E.D.

Theorem 3.2. The Double Cesàro transformation fails to contain includes $D_{p_n, \alpha_n, q_n, \beta_n}$ if there exists an Pringsheim increasing sequence double sequence $[\alpha_{k,l}]$ of integers whose elements belong to neither $[p_n]$ nor $[q_n]$.

Proof. Let us consider the following

$$\bar{M}_{m,n} = \begin{cases} 0, & \text{if } (m,n) \neq (\alpha_m, \beta_n); m, n = 1, 2, 3, \dots \\ x_{m,n}, & \text{if } (m,n) = (\alpha_m, \beta_n); m, n = 1, 2, 3, \dots \end{cases}$$

where $[x]$ is a P-divergent double sequence. Let $[s_{m,n}]$ be double sequence that is mapped by M into \bar{M} . Condition 3.2, $p_m \neq \alpha_m$, and $q_n \neq \beta_n$ assure us that $D_{p_n, \alpha_n, q_n, \beta_n}$ sum $[x]$ to zero. Since M fails to sum $[x]$. Q.E.D.

The following theorem follows from Theorem 3.1 and 3.2.

Theorem 3.3. The four dimensional summability method M include $D_{p_n, \alpha_n, q_n, \beta_n}$ where p_n and q_n for almost all n is a give non-negative integer p if and only if α_n and β_n are almost all positive integers.

Theorem 3.4. The four dimensional summability method M include $D_{m-1, q_m, n-1, \beta_n}$ where $q_m - m$ and $p_n - n$ both increases monotonically with m and n , respectively if and only if $q_m - m$ and $p_n - n$ both are both bounded.

Proof. To establish to sufficiency part not that $q_m - m$ and $p_n - n$ must have a limit, say α and β , respectively and that $q_m - m = \alpha$ and $p_n - n = \beta$ for almost all m and n . Thus $\{q_m\}$ and $\{p_n\}$ contains almost all positive integers and Theorem 3.1 grants us the results.

To established the necessary part, suppose $q_m - m$ and $p_n - n$ increases monotonically with m and n are both unbounded. The goal now is to show that the set of double sequences that are double Cesàro summable are not summable by the double Deferred Cesàro mean. Let $m_1 = n_1 = 1$ and m_2 and n_2 are the smallest integers such that

$$q_m - m > q_{m_1} - m_1 \text{ and } p_n - n > p_{n_1} - n_1$$

Then choose m_3 and n_3 to be the smallest integers m and n such that

$$q_m - m > q_{m_2} - m_2 \text{ and } p_n - n > p_{n_2} - n_2.$$

Thus having chosen

$$m_1 < m_2 < \cdots < m_\alpha \text{ and } n_1 < n_2 < \cdots < n_\beta.$$

We then choose $m_{\alpha+1}$ and $n_{\beta+1}$ to be the smallest integers such that

$$q_m - m > q_{m_\alpha} - m_\alpha \text{ and } p_n - n > p_{n_\beta} - n_\beta.$$

We then define a double sequence $\{s_{k,l}\}$ as follows:

$$s_{k,l} = \begin{cases} q_{m_i} p_{n_j}, & \text{if } k = q_{m_i} \text{ and } l = p_{n_j}; i, j = 1, 2, 3, \dots \\ kl, & \text{if } k \neq q_{m_i} \text{ and/or } l \neq p_{n_j}; i, j = 1, 2, 3, \dots \end{cases}$$

Note $D_{m,n}$ maps $\{s_{k,l}\}$ into 1. for all (m,n) . Thus $\{s_{k,l}\}$ is D-summable to 1. Also $\{s_{k,l}\}$ is not M-summable, since $P\text{-}\lim_{k,l} \frac{s_{k,l}}{k,l} \neq 0$. Thus the double Cesàro mean is contained in the double Deferred Cesàro mean. Q.E.D.

Theorem 3.5. Let $D_{m-1, q_m, n-1, p_n}$ be a Deferred Cesàro mean with $q_m = m$, $p_n = n$; $m \neq \alpha_1, \alpha_2, \dots$ and $n \neq \beta_1, \beta_2, \dots$ with

$$q_{\alpha_i} = \alpha_{i+1} - 1; i = 1, 2, 3, \dots, \alpha_m$$

and

$$p_{\beta_j} = \beta_{j+1} - 1; j = 1, 2, 3, \dots, \beta_n$$

where $\{q_{\alpha_i}\}$ and $\{p_{\beta_j}\}$ are increasing single dimensional sequences of integers such that $\alpha_m > m$ and $\beta_n > n$. Then D is included in M if and only if $\frac{q_m}{m}$ and $\frac{p_n}{n}$ are bounded for all m and n .

Proof. Note $D_{m-1, m, n-1, n}$ is the identity transformation. Let us consider the ordered pair (m, n) and observe that for each pair (m, n) , let

$$i = i_m \text{ and } j = j_n$$

be such that $\alpha_i \leq m < \alpha_{i+1}$ and $\beta_j \leq n < \beta_{j+1}$. Let $\{s_{m,n}\}$ be a given double sequence and consider the transformation

$$M_{m,n} = \frac{1}{mn} \begin{bmatrix} s_{1,1} + s_{1,2} + s_{1,3} + \cdots + s_{1,n} \\ s_{2,1} + s_{2,2} + s_{2,3} + \cdots + s_{2,n} \\ \vdots \\ s_{m,1} + s_{m,2} + s_{m,3} + \cdots + s_{m,n} \end{bmatrix}.$$

Using the definition of double Deferred Cesàro mean we obtain the following

$$\begin{aligned}
 & \left[\begin{array}{cccc} s_{1,1} & + & \cdots & + & s_{1,\beta_1-1} \\ s_{2,1} & + & \cdots & + & s_{2,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_1-1,1} & + & \cdots & + & s_{\alpha_1-1,\beta_1-1} \\ s_{\alpha_1,1} & + & \cdots & + & s_{\alpha_1,\beta_1-1} \\ s_{\alpha_1+1,1} & + & \cdots & + & s_{\alpha_1+1,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_2-1,1} & + & \cdots & + & s_{\alpha_2-1,\beta_1-1} \\ s_{\alpha_2,1} & + & \cdots & + & s_{\alpha_2,\beta_1-1} \\ s_{\alpha_2+1,1} & + & \cdots & + & s_{\alpha_2+1,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_3-1,1} & + & \cdots & + & s_{\alpha_3-1,\beta_1-1} \end{array} \right] + \cdots + \left[\begin{array}{cccc} s_{1,\beta_j} & + & \cdots & + & s_{1,\beta_{j+1}-1} \\ s_{2,1} & + & \cdots & + & s_{2,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_1-1,\beta_j} & + & \cdots & + & s_{\alpha_1-1,\beta_{j+1}-1} \\ s_{\alpha_1,\beta_j} & + & \cdots & + & s_{\alpha_1,\beta_{j+1}-1} \\ s_{\alpha_1+1,1} & + & \cdots & + & s_{\alpha_1+1,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_2-1,\beta_j} & + & \cdots & + & s_{\alpha_2-1,\beta_{j+1}-1} \\ s_{\alpha_2,\beta_j} & + & \cdots & + & s_{\alpha_2,\beta_{j+1}-1} \\ s_{\alpha_2+1,1} & + & \cdots & + & s_{\alpha_2+1,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_3-1,\beta_j} & + & \cdots & + & s_{\alpha_3-1,\beta_{j+1}-1} \end{array} \right] \\
 & \vdots \\
 & \left[\begin{array}{cccc} s_{\alpha_i,1} & + & \cdots & + & s_{\alpha_i,\beta_1-1} \\ s_{\alpha_i+1,1} & + & \cdots & + & s_{\alpha_i+1,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_{i+1}-1,1} & + & \cdots & + & s_{\alpha_{i+1}-1,\beta_1-1} \end{array} \right] + \cdots + \left[\begin{array}{cccc} s_{\alpha_i,\beta_j} & + & \cdots & + & s_{\alpha_i,\beta_{j+1}-1} \\ s_{\alpha_i+1,1} & + & \cdots & + & s_{\alpha_i+1,\beta_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_{i+1}-1,\beta_j} & + & \cdots & + & s_{\alpha_{i+1}-1,\beta_{j+1}-1} \end{array} \right]
 \end{aligned}$$

Let us denote the above sum by $\Omega_{m,n}$ and the sum below by $\Lambda_{m,n}$

$$\begin{aligned}
 & \begin{array}{cccc} s_{1,n+1} & + & \cdots & + & s_{1,\beta_{j+1}-1} \\ s_{2,n+1} & + & \cdots & + & s_{2,\beta_{j+1}-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{m,n+1} & + & \cdots & + & s_{m,\beta_{j+1}-1} \\ s_{m+1,1} & + & \cdots & + & s_{m+1,n+1} & + & s_{m+1,n+1} & + & \cdots & + & s_{m+1,\beta_{j+1}-1} \\ s_{m+2,1} & + & \cdots & + & s_{m+2,n+1} & + & s_{m+2,n+1} & + & \cdots & + & s_{m+2,\beta_{j+1}-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{\alpha_{i+1}-1,1} & + & \cdots & + & s_{\alpha_{i+1}-1,n+1} & + & s_{\alpha_{i+1}-1,n+1} & + & \cdots & + & s_{\alpha_{i+1}-1,\beta_{j+1}-1} \end{array}
 \end{aligned}$$

Therefore $M_{m,n} = \frac{1}{mn}(\Omega_{m,n} - \Lambda_{m,n})$. It is important to observe that if $m = \alpha_{i+1} - 1$ and/or $n = \beta_{j+1} - 1$ then the terms in $\Lambda_{m,n}$ will not exist that is if $m = \alpha_{i+1} - 1$ and/or $n = \beta_{j+1} - 1$ then the terms in the rows and/or columns will not exist. Let us also denote the following sum by

$\bar{\Omega}_{m,n}$

$$\begin{array}{ccccccc}
\frac{\sum_{k,l=1,1}^{\alpha_1, \beta_1} s_{k,l}}{mn} & + & \frac{\alpha_1(\beta_2 - \beta_1)}{mn} D_{0,1} & + & \cdots & + & \frac{\alpha_1(\beta_{j+1} - \beta_j)}{mn} D_{0,j} \\
\frac{(\alpha_2 - \alpha_1)\beta_1}{mn} D_{1,0} & + & \frac{(\alpha_2 - \alpha_1)(\beta_2 - \beta_1)}{mn} D_{1,1} & + & \cdots & + & \frac{(\alpha_2 - \alpha_1)(\beta_{j+1} - \beta_j)}{mn} D_{1,j} \\
\frac{(\alpha_3 - \alpha_2)\beta_1}{mn} D_{2,0} & + & \frac{(\alpha_3 - \alpha_2)(\beta_2 - \beta_1)}{mn} D_{2,1} & + & \cdots & + & \frac{(\alpha_3 - \alpha_2)(\beta_{j+1} - \beta_j)}{mn} D_{2,j} \\
+ & + & + & + & + & + & + \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
+ & + & + & + & + & + & + \\
\frac{(\alpha_{i+1} - \alpha_i)\beta_1}{mn} D_{i,0} & + & \frac{(\alpha_{i+1} - \alpha_i)(\beta_2 - \beta_1)}{mn} D_{i,1} & + & \cdots & + & \frac{(\alpha_{i+1} - \alpha_i)(\beta_{j+1} - \beta_j)}{mn} D_{i,j}
\end{array}$$

and also denote the following sum by $\bar{\Lambda}_{m,n}$

$$\begin{array}{ccccccc}
& & & & D_{1,n+1} & + & \cdots & + & D_{1,\beta_{j+1}-1} \\
& & & & D_{2,n+1} & + & \cdots & + & D_{2,\beta_{j+1}-1} \\
& & & & \vdots & + & \cdots & + & \vdots \\
& & & & D_{m,n+1} & + & \cdots & + & D_{m,\beta_{j+1}-1} \\
D_{m+1,1} & + & \cdots & + & D_{m+1,n} & + & D_{m+1,n+1} & + & \cdots & + & D_{m+1,\beta_{j+1}-1} \\
D_{m+2,1} & + & \cdots & + & D_{m+2,n} & + & D_{m+2,n+1} & + & \cdots & + & D_{m+2,\beta_{j+1}-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
D_{\alpha_{i+1}-1,1} & + & \cdots & + & D_{\alpha_{i+1}-1,n} & + & D_{\alpha_{i+1}-1,n+1} & + & \cdots & + & D_{\alpha_{i+1}-1,\beta_{j+1}-1}
\end{array}$$

Then we can now rewrite $M_{m,n}$ in the following manner $\bar{\Omega}_{m,n} - \frac{1}{mn}\bar{\Lambda}_{m,n}$. The relation $\bar{\Omega}_{m,n} - \frac{1}{mn}\bar{\Lambda}_{m,n}$ hold for each (m,n) and defines a four-dimensional transformation of the form

$$\sigma_{m,n} = \sum_{k,l=1,1}^{\infty, \infty} a_{m,n,k,l} s_{k,l}$$

which carries $D_{m,n}$ into $M_{m,n}$. This transformation clearly satisfies RH_1 and RH_2 . This transformation satisfies RH_3 and RH_4 only if $\frac{2\alpha_{i+1}-m-2}{m}$ and $\frac{2\beta_{j+1}-n-2}{n}$ are bounded respectively for each (m,n) , which is equivalent to $\frac{\alpha_{i+1}}{m}$ and $\frac{\alpha_{i+1}}{m}$ are bounded, which is also equivalent to the boundedness of $\frac{q_m}{m}$ and $\frac{p_n}{n}$ for each (m,n) . Condition RH_5 and RH_6 hold only when both $\frac{2\alpha_{i+1}-m-2}{m}$ and $\frac{2\beta_{j+1}-n-2}{n}$ are bounded, and as above the is equivalent to boundedness of $\frac{q_m}{m}$ and $\frac{p_n}{n}$ for each (m,n) . Since D is a factorable four-dimensional summability matrix the main theorem in [1] assure us that it has an inverse. Thus the result follows for the Robison-Hamilton characterization of regularity. Q.E.D.

Theorem 3.6. The double Cesàro mean includes $D_{m-1,q_m,n-1,p_n}$ be a Deferred Cesàro mean with $q_m = m, p_n = n; m \neq \alpha_1, \alpha_2, \dots$ and $n \neq \beta_1, \beta_2, \dots$ with

$$q_{\alpha_i} = \alpha_{i+1} - 1; i = 1, 2, 3, \dots, \alpha_m$$

and

$$p_{\beta_j} = \beta_{j+1} - 1; j = 1, 2, 3, \dots, \beta_n$$

where $\{q_{\alpha_i}\}$ and $\{p_{\beta_j}\}$ are increasing single dimensional sequences of integers such that $\alpha_m > m$ and $\beta_n > n$.

Proof. Observe that for each pair (m, n) , let

$$i = i_m \text{ and } j = j_n$$

be such that $h_i \leq m < h_{i+1}$ and $t_j \leq n < t_{j+1}$. Let $\{s_{m,n}\}$ be a given double sequence and consider the following four dimensional Cesàro transformation

$$M_{m,n} = \frac{1}{mn} \begin{bmatrix} s_{1,1} + s_{1,2} + s_{1,3} + \cdots + s_{1,n} \\ s_{2,1} + s_{2,2} + s_{2,3} + \cdots + s_{2,n} \\ \vdots \\ s_{m,1} + s_{m,2} + s_{m,3} + \cdots + s_{m,n} \end{bmatrix}.$$

Using the definition of double Deferred Cesàro mean we can rewrite $mnM_{m,n}$ using the following, respectively, $A_{m,n}^i, A_{m,n}^{i-1}, A_{m,n}^{i-2}, \dots, A_{m,n}^\alpha$ and $K_{m,n}$ where $K_{m,n}$ is

$$\begin{matrix} s_{1,1} & + & s_{1,2} & + & \cdots & + & s_{1,\beta\delta-1} \\ s_{2,1} & + & s_{2,2} & + & \cdots & + & s_{2,\beta\delta-1} \\ \vdots & & \vdots & & \cdots & & \vdots \\ s_{\beta\Delta-1,1} & + & s_{\beta\Delta-1,2} & + & \cdots & + & s_{\beta\Delta-1,\beta\delta-1} \end{matrix}$$

with Δ and δ are 2 or 1 depending on weather α and/ or β are odd or even, and the A 's are define below, respectively

$$\begin{matrix} s_{m,n} & + & s_{m,n-1} & + & \cdots & + & s_{m,t_j+1} \\ s_{m-1,n} & + & s_{m-1,n-1} & + & \cdots & + & s_{m-1,t_j+1} \\ \vdots & + & \vdots & + & \cdots & + & \vdots \\ s_{h_i+1,n} & + & s_{h_i+1,n-1} & + & \cdots & + & s_{h_i+1,t_j+1} \end{matrix},$$

$$\begin{matrix} s_{h_i,n} & + & s_{h_i,n-1} & + & \cdots & + & s_{h_i,t_j+1} & + & s_{\alpha_i,t_j} & + & \cdots & + & s_{h_i,t_j-1} \\ \vdots & + & \vdots & + & \cdots & + & \vdots & \vdots & \vdots & + & \cdots & + & \vdots \\ s_{h_{i-1},n} & + & s_{h_{i-1},n-1} & + & \cdots & + & s_{h_{i-1},t_j+1} & + & s_{h_{i-1},t_j} & + & \cdots & + & s_{h_{i-1},t_j-1} \end{matrix},$$

$$\begin{matrix} s_{h_{i-1}-1,n} & + & s_{h_{i-1}-1,n-1} & + & \cdots & + & s_{h_{i-1}-1,t_j+1} & + & s_{h_{i-1}-1,t_j} & + & \cdots & + & s_{h_{i-1}-1,t_j-2+1} \\ \vdots & + & \vdots & + & \cdots & + & \vdots & \vdots & \vdots & + & \cdots & + & \vdots \\ s_{h_{i-2}+1,n} & + & s_{h_{i-2}+1,n-1} & + & \cdots & + & s_{h_{i-2}+1,t_j+1} & + & s_{h_{i-2}+1,t_j} & + & \cdots & + & s_{h_{i-2}+1,t_j-2+1} \end{matrix},$$

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